The Representation Theory of Co-triangular Semisimple Hopf Algebras

Pavel Etingof
Department of Mathematics
Harvard University
Cambridge, MA 02138

Shlomo Gelaki
Department of Mathematics
University of Southern California
Los Angeles, CA 90089

December 18, 1998

1 Introduction

In [EG1, Theorem 2.1] we prove that *any* semisimple triangular Hopf algebra A over an algebraically closed field of characteristic 0 (say the field of complex numbers $\mathbf{C}$) is obtained from a finite group after twisting the ordinary comultiplication of its group algebra in the sense of Drinfeld [D]; that is $A = \mathbf{C}[G]^J$ for some finite group G and a twist $J \in \mathbf{C}[G] \otimes \mathbf{C}[G]$. In [EG2] we show how to construct twists for certain solvable non-abelian groups by iterating twists of their abelian subgroups, and thus obtain new non-trivial semisimple triangular Hopf algebras. We also show how any non-abelian finite group which admits a bijective 1-cocycle with coefficients in an abelian group, gives rise to a non-trivial semisimple minimal triangular Hopf algebra. Such non-abelian groups (which are necessarily solvable [ESS]) exist in abundance and were constructed in [ESS] in connection with set-theoretical solutions to the quantum Yang-Baxter equation.

If A is minimal triangular then A and A^{*op} are isomorphic as Hopf algebras. But any non-trivial semisimple triangular A which is not minimal, gives rise to a new Hopf algebra A^*, which is also semisimple by [LR]. These are very interesting semisimple Hopf algebras which arise from finite groups, and they are abundant by the constructions given in [EG2]. Generally, the dual Hopf algebra of a triangular Hopf algebra is called *co-triangular* in the literature.

In this paper we explicitly describe the representation theory of co-triangular semisimple Hopf algebras $A^* = (\mathbf{C}[G]^J)^*$ in terms of representations of some associated groups. As a corollary we prove that Kaplansky's 6th conjecture [K] holds for A^*; that is that the dimension of any irreducible representation of A^* divides the dimension of A.

We note that we have used in an essential way the results of the paper [Mo], from which we learned a great deal.

2 Preliminaries

2.1 Projective Representations and Central Extensions

Here we recall some basic facts about projective representations and central extensions. They can be found in textbooks, e.g. [CR, Section 11E].

A projective representation of a group Γ is a vector space V together with a homomorphism of groups $\pi_V : \Gamma \to PGL(V)$, where $PGL(V) \cong GL(V)/\mathbf{C}$ is the projective linear group.

A linearization of a projective representation V of Γ is a central extension $\hat{\Gamma}$ of Γ by a central subgroup ζ together with a linear representation $\tilde{\pi}_V : \hat{\Gamma} \to GL(V)$ which descends to π_V. If V is a finite-dimensional projective representation of Γ then there exists a linearization of V such that ζ is finite (in fact, one can make $\zeta = \mathbb{Z}/(\dim V)\mathbb{Z}$).

Any projective representation V of Γ canonically defines a cohomology class $[V] \in H^2(\Gamma, \mathbf{C}^*)$. The representation V can be lifted to a linear representation of Γ if and only if $[V] = 0$.

2.2 The Algebras Associated With a Twist

Let H be a finite group, and let $J \in \mathbf{C}[H] \otimes \mathbf{C}[H]$ be a *minimal* twist (see [EG1]). That is, the right (and left) components of the R-matrix $R = J_{21}^{-1} J$ span $\mathbf{C}[H]$. Define two coalgebras $(A_1, \Delta_1, \varepsilon), (A_2, \Delta_2, \varepsilon)$ as follows: $A_1 = A_2 = \mathbf{C}[H]$ as vector spaces, the coproducts are determined by

$$\Delta_1(x) = (x \otimes x)J, \ \Delta_2(x) = J^{-1}(x \otimes x)$$

for all $x \in H$, and ε is the ordinary counit of $\mathbf{C}[H]$. Note that since J is a twist, Δ_1 and Δ_2 are indeed coassociative. Clearly the dual algebras A_1^* and A_2^* are spanned by $\{\delta_h | h \in H\}$, where $\delta_h(h') = \delta_{hh'}$.

Theorem 2.2.1 *Let A_1^* and A_2^* be as above. The following hold:*

1. *A_1^* and A_2^* are $H-$algebras via*

$$\rho_1(h)\delta_y = \delta_{hy}, \quad \rho_2(h)\delta_y = \delta_{yh^{-1}}$$

   respectively.

2. *$A_1^* \cong A_2^{*op}$ as $H-$algebras (where H acts on A_2^{*op} as it does on A_2^*).*

3. *The algebras A_1^* and A_2^* are simple, and are isomorphic as $H-$modules to the regular representation R_H of H.*

Proof: The proof of part 1 is straightforward.

The proof of part 3 follows from the results in [Mo]. Namely, it follows from [Mo, Proposition 14] that in the case of a minimal twist the group St defined in [Mo] (which is, by definition, a subgroup of H) coincides with H. Therefore, by [Mo, Propositions 6,7] the algebras A_1^* and A_2^* are simple. Furthermore, by [Mo, Propositions 11,12], the actions of H on A_1^* and A_2^* are isomorphic to the regular representation of H.

Let us prove part 2. Let S_0, Δ_0, m_0 denote the standard antipode, coproduct and multiplication of $\mathbf{C}[H]$, and define $Q = m_0(S_0 \otimes I)(J)$. Then it is straightforward to verify that Q is invertible, and $(S_0 \otimes S_0)(J) = (Q \otimes Q)J_{21}^{-1}\Delta_0(Q)^{-1}$ (see e.g. (2.17) in [Ma, Section 2.3]). Hence the map $A_2^* \to A_1^{*op}$, $\delta_x \mapsto \delta_{S_0(x)Q^{-1}}$ determines an $H-$algebra isomorphism. ■

Corollary 2.2.2 *Let A be a semisimple minimal triangular Hopf algebra over $\mathbf{C}$ with Drinfeld element u. If $u = 1$ then $dimA$ is a square, and if $u \neq 1$ then $2dimA$ or $dimA$ is a square.*

Proof: The first statement follows from [EG1, Theorem 2.1] and part 3 of Theorem 2.2.1. To prove the second statement, let (A, R, u) be a semisimple minimal triangular Hopf algebra with $u \neq 1$, and (A, R', u') be obtained from (A, R, u) by changing R so that the new Drinfeld element $u' = 1$ (as in [EG1]). Then $(A, R') = (\mathbf{C}[H]^J, J_{21}^{-1}J)$ for some finite group H. Let $A_{min} = \mathbf{C}[H']^J$ be the minimal triangular Hopf subalgebra of (A, R') where $H' \subset H$ is a subgroup, and J is a minimal twist for H'. It is clear that H is generated by H' and u. Since u is a central grouplike element of order 2, we get that the index of H' is at most 2. This implies our statement. ■

Let A_1^*, A_2^* be the H-algebras as in Theorem 2.2.1. Since the algebras A_1^*, A_2^* are simple, the actions of H on A_1^*, A_2^* give rise to projective representations $H \to PGL(|H|^{1/2}, \mathbf{C})$. We will denote these projective representations by V_1, V_2 (they can be thought of as the simple modules over A_1^*, A_2^*, with the induced projective action of H). Note that part 2 of Theorem 2.2.1 implies that V_1, V_2 are dual to each other, hence that $[V_1] = -[V_2]$.

3 The Main Result

Let (A, R) be a semisimple triangular Hopf algebra over $\mathbf{C}$, and assume that the Drinfeld element u is 1 (this can be always achieved by a simple modification of R, without changing the Hopf algebra structure [EG1]). Then by [EG1, Theorem 2.1], there exist finite groups $H \subset G$ and a minimal twist $J \in \mathbf{C}[H] \otimes \mathbf{C}[H]$ such that $(A, R) \cong (\mathbf{C}[G]^J, J_{21}^{-1}J)$ as triangular Hopf algebras. So from now on we will assume that A is of this form.

Consider the dual Hopf algebra A^*. It has a basis of δ-functions δ_g. The first simple but important fact about the structure of A^* as an algebra is:

Proposition 3.1 *Let Z be a double coset of H in G, and $A_Z^* = \oplus_{g \in Z} \mathbf{C}\delta_g \subset A^*$. Then A_Z^* is a subalgebra of A^*, and $A^* = \oplus_Z A_Z^*$ as algebras.*

Proof: Straightforward. ∎

Thus, to study the representation theory of A^*, it is sufficient to describe the representations of A^*_Z for any Z.

Let Z be a double coset of H in G, and let $g \in Z$. Let $K_g = H \cap gHg^{-1}$, and define the embeddings $\theta_1, \theta_2 : K_g \to H$ given by $\theta_1(a) = g^{-1}ag$, $\theta_2(a) = a$. Denote by W_i the pullback of the projective H-representation V_i to K_g by means of θ_i, $i = 1, 2$.

Our main result is the following theorem, which is proved in the next section.

Theorem 3.2 *Let W_1, W_2 be as above, and let $(\hat{K}_g, \tilde{\pi}_W)$ be any linearization of the projective representation $W = W_1 \otimes W_2$ of K_g. Let ζ be the kernel of the projection $\hat{K}_g \to K_g$, and $\chi : \zeta \to \mathbf{C}^*$ be the character by which ζ acts in W. Then there exists a 1-1 correspondence between isomorphism classes of irreducible representations of A^*_Z and isomorphism classes of irreducible representations of $\hat{K}_g$ with ζ acting by χ. If a representation Y of A^*_Z corresponds to a representation X of $\hat{K}_g$ then $dimY = \frac{|H|}{|K_g|} dimX$.*

As a corollary we get Kaplansky's 6th conjecture [K] for semisimple co-triangular Hopf algebras.

Corollary 3.3 *The dimension of any irreducible representation of a semisimple co-triangular Hopf algebra divides the dimension of the Hopf algebra.*

Proof: Since $dimX$ divides $|K_g|$ (see e.g. [CR, Proposition 11.44]), we have that $\frac{|G|}{\frac{|H|}{|K_g|} dimX} = \frac{|G|}{|H|} \frac{|K_g|}{dimX}$ and the result follows. ∎

In some cases the classification of representations of A^*_Z is even simpler. Namely, let $\overline{g} \in Aut(K_g)$ be given by $a \mapsto g^{-1}ag$. Then we have:

Corollary 3.4 *If the cohomology class $[W_1]$ is $\overline{g}-$invariant then irreducible representations of A^*_Z correspond in a 1-1 manner to irreducible representations of K_g, and if Y corresponds to X then $dimY = \frac{|H|}{|K_g|} dimX$.*

Proof: For any $\alpha \in Aut(K_g)$ and $f \in Hom((K_g)^n, \mathbf{C}^*)$, let $\alpha \circ f \in Hom((K_g)^n, \mathbf{C}^*)$ be given by $(\alpha \circ f)(h_1, \ldots, h_n) = f(\alpha(h_1), \ldots, \alpha(h_n))$ (which determines the action of α on $H^i(K_g, \mathbf{C}^*)$). Then it follows from the identity $[V_1] = -[V_2]$, given at the end of Section 2, that $[W_1] = -\overline{g} \circ [W_2]$. Thus, in our situation $[W] = 0$, hence W comes from a linear representation of K_g. Thus, we can set $\hat{K}_g = K_g$ in the theorem, and the result follows. ∎

Example 3.5 Let $p > 2$ be a prime number, and $H = (\mathbb{Z}/p\mathbb{Z})^2$ with the standard symplectic form $(,) : H \times H \to \mathbf{C}^*$ given by $((x, y), (x', y')) = e^{2\pi i(xy' - yx')/p}$. Then the element

$J = p^{-2} \sum_{a,b \in H} (a,b) a \otimes b$ is a minimal twist for $\mathbf{C}[H]$. Let $g \in GL_2(\mathbb{Z}/p\mathbb{Z})$ be an automorphism of H, and G_0 be the cyclic group generated by g. Let G be the semidirect product of G_0 and H. It is easy to see that in this case, the double cosets are ordinary cosets $g^k H$, and $K_{g^k} = H$. Moreover, one can show either explicitly or using [Mo, Proposition 9], that $[W_1]$ is a generator of $H^2(H, \mathbf{C}^*)$ which is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. The element g^k acts on $[W_1]$ by multiplication by $det(g^k)$. Therefore, by Corollary 3.4, the algebra $A^*_{g^k H}$ has p^2 1-dimensional representations (corresponding to linear representations of H) if $det(g^k) = 1$.

However, if $det(g^k) \neq 1$, then $[W]$ generates $H^2(H, \mathbf{C}^*)$. Thus, W comes from a linear representation of the Heisenberg group $\hat{H}$ (a central extension of H by $\mathbb{Z}/p\mathbb{Z}$) with some central character χ. Thus, $A^*_{g^k H}$ has one p-dimensional irreducible representation, corresponding to the unique irreducible representation of $\hat{H}$ with central character χ (which is W).

4 Proof of Theorem 3.2

Let $Z \subset G$ be a double coset of H in G, and let A_1, A_2 be as in Subsection 2.2. For any $g \in Z$ define the linear map

$$F_g : A^*_Z \to A^*_2 \otimes A^*_1, \quad \delta_y \mapsto \sum_{h,h' \in H : y = hgh'} \delta_h \otimes \delta_{h'}.$$

Proposition 4.1 *Let ρ_1, ρ_2 be as in Theorem 2.2.1. Then:*

1. *The map F_g is an injective homomorphism of algebras.*
2. *$F_{aga'}(\varphi) = (\rho_2(a) \otimes \rho_1(a')^{-1}) F_g(\varphi)$ for any $a, a' \in H$, $\varphi \in A^*_Z$.*

Proof: 1. It is straightforward to verify that the map $F^*_g : A_2 \otimes A_1 \to A_Z$ is determined by $h \otimes h' \mapsto hgh'$, and that it is a surjective homomorphism of coalgebras. Hence the result follows.

2. Straightforward. ∎

For any $a \in K_g$ define $\rho(a) \in Aut(A^*_2 \otimes A^*_1)$ by $\rho(a) = \rho_2(a) \otimes \rho_1(a^g)$, where $a^g = g^{-1} a g$ and ρ_1, ρ_2 are as in Theorem 2.2.1. Then ρ is an action of K_g on $A^*_2 \otimes A^*_1$.

Proposition 4.2 *Let $U_g = (A^*_2 \otimes A^*_1)^{\rho(K_g)}$ be the algebra of invariants. Then $Im(F_g) = U_g$, so $A^*_Z \cong U_g$ as algebras.*

Proof: It follows from Proposition 4.1 that $Im(F_g) \subseteq U_g$, and $rk(F_g) = dim A^*_Z = \dfrac{|H|^2}{|K_g|}$. On the other hand, by Theorem 2.2.1, A^*_1, A^*_2 are isomorphic to the regular representation R_H

of H. Thus, A_1^*, A_2^* are isomorphic to $\frac{|H|}{|K_g|}R_{K_g}$ as representations of K_g, via $\rho_1(a), \rho_2(a^g)$. Thus, $A_2^* \otimes A_1^* \cong \frac{|H|^2}{|K_g|^2}(R_{K_g} \otimes R_{K_g}) \cong \frac{|H|^2}{|K_g|}R_{K_g}$. So U_g has dimension $|H|^2/|K_g|$, and the result follows. ∎

Now we are in a position to prove Theorem 3.2. Since $W_i \otimes W_i^* \cong A_i^*$ for $i = 1, 2$, it follows from Theorem 2.2.1 that $W_1 \otimes W_2 \otimes W_1^* \otimes W_2^* \cong \frac{|H|^2}{|K_g|}R_{K_g}$ as $\hat{K}_g$ modules. Thus, if χ_W is the character of $W = W_1 \otimes W_2$ as a $\hat{K}_g$ module then

$$|\chi_W(x)|^2 = 0,\ x \notin \zeta \ \ and \ \ |\chi_W(x)|^2 = |H|^2,\ x \in \zeta.$$

Therefore,

$$\chi_W(x) = 0,\ x \notin \zeta \ \ and \ \ \chi_W(x) = |H| \cdot x_W,\ x \in \zeta,$$

where x_W is the root of unity by which x acts in W. Now, it is clear from the definition of U_g (see Proposition 4.2) that $U_g = End_{\hat{K}_g}(W)$. Thus if $W = \bigoplus_{M \in Irr(\hat{K}_g)} W(M) \otimes M$, where $W(M) = Hom_{\hat{K}_g}(M, W)$ is the multiplicity space, then $U_g = \bigoplus_{M : W(M) \neq 0} End_{\mathbf{C}}(W(M))$. So $\{W(M) | W(M) \neq 0\}$ are the irreducible representations of U_g. Thus the following implies the theorem:

Lemma.

1. $W(M) \neq 0$ if and only if for all $x \in \zeta$, $x_{|M} = x_{|W}$.
2. If $W(M) \neq 0$ then $dimW(M) = \frac{|H|}{|K_g|}dimM$.

Proof of the Lemma. The "only if" part of 1 is clear. For the "if" part compute $dimW(M)$ as the inner product (χ_W, χ_M). We have

$$(\chi_W, \chi_M) = \sum_{x \in \zeta} \frac{|H|}{|\hat{K}_g|} x_{|W} \cdot dimM \cdot \bar{x}_{|M}.$$

If $x_{|M} = x_{|W}$ then

$$(\chi_W, \chi_M) = \sum_{x \in \zeta} \frac{|H|}{|\hat{K}_g|} dimM = \frac{|H||\zeta|}{|\hat{K}_g|} dimM = \frac{|H|}{|K_g|} dimM.$$

This proves part 2 as well, and hence concludes the proof of the theorem. ∎

References

[CR] C. Curtis and I. Reiner, Methods of Representation Theory **1**, John Wiley & Sons, Inc. (1981).

[D] V. Drinfeld, On Almost Cocommutative Hopf Algebras, *Leningrad Math. J* **1** (1990), 321-342.

[EG1] P. Etingof and S. Gelaki, Some Properties of Finite-Dimensional Semisimple Hopf Algebras, *Mathematical Research Letters* **5** (1998), 191-197.

[EG2] P. Etingof and S. Gelaki, A Method of Construction of Finite-Dimensional Triangular Semisimple Hopf Algebras, *Mathematical Research Letters* **5** (1998), 551-561.

[ESS] P. Etingof, T. Schedler and A. Soloviev, Set-Theoretic Solutions to the Quantum Yang-Baxter Equation, *Duke Math. J.*, to appear, math.QA/9801047.

[K] I. Kaplansky, Bialgebras, University of Chicago, 1975.

[LR] R. G. Larson and D. E. Radford, Finite-Dimensional Cosemisimple Hopf Algebras in Characteristic 0 are Semisimple, *J. Algebra* **117** (1988), 267-289.

[Ma] S. Majid, Foundations of quantum group theory, Cambridge University Press, 1995.

[Mo] M. Movshev, Twisting in group algebras of finite groups, *Func. Anal. Appl.* **27** (1994), 240-244.